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Stochastic Ornstein–Uhlenbeck Capacitors

Iddo Eliazar[1] and Joseph Klafter[2]

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We introduce and study a class of random capacitor systems which are both charged and discharged stochastically. A capacitor is 'fed' by a random inflow with stationary and independent increments. Discharging occurs according to a Markovian rate which is linear in the capacitor's level. The resulting capacitor dynamics are Markovian, stochastically cyclic, and regenerative. We coin these systems "Lévy-charged Ornstein–Uhlenbeck capacitors". Various random quantities associated with these systems are analyzed, including: the time-to-discharge; the duration of the charging cycle; the trajectory and the peak height of the capacitor level during a charging cycle; and, the capacitor's stationary equilibrium level. Furthermore, we show that there are sharp distinctions between these capacitor systems and corresponding 'standard' Lévy-driven Ornstein–Uhlenbeck systems.

KEY WORDS: Stochastic capacitors; Lévy inflow; Ornstein–Uhlenbeck dynamics.

1. INTRODUCTION

In this manuscript, we introduce and analyze a theoretical model of random capacitor systems where both their charging and discharging mechanisms are stochastic. Let us first describe the model and then specify possible applications.

Consider a stochastic capacitor 'fed' continuously by a random inflow of charge. We define three processes – the capacitor process $X = (X(t))_{t\geqslant 0}$, the charging process $L = (L(t))_{t\geqslant 0}$, and the discharge process $D = (D(t))_{t\geqslant 0}$ – as follows

[1]Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel; e-mail: eliazar@math.biu.ac.il, eliazar@post.tau.ac.il
[2]School of Chemistry, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel; e-mail: klafter@post.tau.ac.il

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- *Capacitor process*: $X(t)$ is the level, at time t, of the capacitor. The capacitor process X is non-negative valued.

- *Charging process*: $L(t)$ is the cumulative charge inflow, 'fed' into the capacitor, during the time period $[0, t]$. The increments of the charging process L are non-negative valued.

- *Discharge process*: $D(t)$ is the number of capacitor discharges that have occurred during the time period $[0, t]$. The discharge process D is a counting process: it is integer-valued, non-decreasing, and has unit-size jumps.

All three processes are assumed, with no loss of generality, to start at the origin (i.e., $X(0) = L(0) = D(0) = 0$). The stochastic dynamics of the capacitor process X are given by

$$X(dt) = -X(t)D(dt) + L(dt). \tag{1}$$

That is: (a) if there is no discharge during the infinitesimal time interval $(t, t + dt)$ then the capacitor level increases by the value $L(dt)$ – the charge inflow during this time interval; (b) if the capacitor discharges during the infinitesimal time interval $(t, t + dt)$ then the capacitor level reduces from the (pre-discharge) level $X(t)$ to the (post-discharge) level 0.

Rather than exploring 'standard' deterministic capacitors – i.e., capacitors with fixed threshold levels – we consider the more complicated case of *stochastic* capacitors, where discharges occur randomly. To illustrate random discharges consider the example of a house-of-cards. A house-of-cards has no *a-priori* threshold level. Rather, as cards are added to the house its intrinsic probability of crashing grows larger and larger – until the inevitable tumble-down finally occurs. Specifically, we assume the random discharging mechanism to be governed by a *Markovian rate function* $r(x)$: if the capacitor's level at time t is x then the probability of discharging during the infinitesimal time interval $(t, t + dt)$ is $r(x)dt$. This implies that the discharge process D is given by

$$D(t) = \Pi\left(\int_0^t r(X(s))ds\right), \tag{2}$$

where $\Pi = (\Pi(t))_{t \geqslant 0}$ is a standard *Poisson* process (i.e., a Poisson process with unit rate), which is *independent* of the charging process L.

Hence, the stochastic dynamics of a random capacitor system are given by the pair of Eqs. (1) and (2). The 'noisy inputs' to the system are the charging process L and the Poisson process Π – which is the 'trigger' of the discharges. These noisy inputs are independent, but their combined influence on the resulting capacitor process X is highly convoluted due to the strong coupling of Eqs. (1) and (2). Presented alternatively, the stochastic dynamics of the capacitor process X are given by the following transition probabilities

$$X(t) \longmapsto \begin{cases} X(t)+L(dt) & \text{with prob. } 1-r(X(t))dt, \\ 0 & \text{with prob. } \quad r(X(t))dt. \end{cases} \tag{3}$$

Possible applications of the random capacitor model are systems exhibiting stochastic growth-collapse evolutionary cycles. That is, systems whose temporal evolution is governed by stochastic growth-collapse cycles: a period of steady growth followed by an abrupt event collapsing/re-setting the system back to its 'ground level' (immediately after which a new growth-collapse cycle commences). Examples include: the house-of-cards described above; sandpile-type models and systems in self-organized criticality (see ref. 1 and references therein); stick–slip models of interfacial friction;[2] Burridge–Knopoff type models of earthquakes and continental drift;[3] and, 'gazing models' of human eye movement.[4] See also ref. 5 for a related study of systems exhibiting stochastic growth-collapse behavior.

The manuscript is organized as follows; in Section 2 we introduce the notion of Lévy-charged Ornstein–Uhlenbeck capacitor systems, to be studied and analyzed in subsequent sections. In Section 3 we study the distribution of the 'time-to-discharge'. Namely; how long would it take a capacitor system, initiated from the level $x>0$, to discharge? In Section 4 we analyze the peaks of the capacitor process, deriving their Laplace transform, mean, and – in the case of heavy-tailed systems – their tail behavior. In Section 5 we study the trajectory of the capacitor process along a charging cycle, *conditioned* on the information that the cycle is of specific length. We conclude, in Section 6, with an analysis of the evolution and stationary equilibrium behavior of the capacitor process X.

A note about notations

Throughout the manuscript: $\mathbf{P}(\cdot)$ is the probability; $\mathbf{E}[\cdot]$ is the expectation; $\mathbf{P}(\cdot|\mathcal{E})$ is the conditional probability with respect to the event/information $\mathcal{E}$; and $\mathbf{E}[\cdot|\mathcal{E}]$ is the conditional probability with respect to the event/information $\mathcal{E}$.

2. LÉVY-CHARGED ORNSTEIN–UHLENBECK CAPACITORS

In this manuscript we study the class of stochastic capacitor systems where

- the charging process L has *stationary* and *independent* increments;
- the discharge rate function is *linear* $r(x) = \lambda x$, $x \geqslant 0$ ($\lambda > 0$).

This class is henceforth referred to as *Lévy-charged Ornstein–Uhlenbeck* capacitors. The term '*Lévy-charged*' is due to the fact that the charging process is a Lévy process (see the explanation below). The term '*Ornstein–Uhlenbeck*' is due to the fact that discharge rate is linear (in the system level). Let us elaborate.

2.1. Charge Inflow

The stationarity and independence of the increments of the charging process L, combined with their non-negativity, implies that the inflow L is a *one-sided Lévy process*, or a *Lévy subordinator*.[6–12]

Being one-sided Lévy, the charging process L admits a Laplace transform of the form ($\omega \geqslant 0$)

$$\mathbf{E}[\exp\{-\omega L(t)\}] = \exp\{-\phi(\omega)t\} \ , \tag{4}$$

where $\phi(\omega)$, $\omega \geqslant 0$, is the process' *characteristic Laplace exponent* (see, e.g., ref. 9). We set $\Phi(\omega)$, $\omega \geqslant 0$, to be the primitive of ϕ

$$\Phi(\omega) = \int_0^{\omega} \phi(u) du \ . \tag{5}$$

The one-dimensional Laplace transform given in Eq. (4) – of the random variable $L(t)$ – extends to the infinite-dimensional Laplace transform of the *entire process* L

$$\mathbf{E}\left[\exp\left\{-\int_0^\infty \varphi(t) L(dt)\right\}\right] = \exp\left\{-\int_0^\infty \phi\left(\varphi(t)\right) dt\right\} \tag{6}$$

for any 'nice' test function $\varphi(t)$, $t \geqslant 0$ (being the infinite-dimensional Laplace coordinate).

We give a few examples of *one-sided Lévy* processes:

1. Deterministic processes: $\phi(\omega) = a\omega$ ($a > 0$) (corresponding to $L(t) \equiv at$).

2. Compound Poisson processes with exponential jumps: $\phi(\omega) = a\omega/(\nu + \omega)$ ($a, \nu > 0$).

3. Gamma processes (Lévy processes with Gamma-distributed increments): $\phi(\omega) = a \ln(1 + \omega/\nu)$ ($a, \nu > 0$).

4. Selfsimilar processes[13] ('fractal' Lévy processes which are invariant under changes of scale): $\phi(\omega) = a\omega^{\alpha}$ $(a > 0,\ 0 < \alpha < 1)$. The increments of these processes are heavy-tailed[3], and have *no* finite moments.

2.2. Capacitor Dynamics

Equations (1) and (2), in the case of a linear discharge rate, yield the following stochastic capacitor dynamics

$$\begin{aligned} X(dt) &= -X(t)D(dt) + L(dt), \\ D(t) &= \Pi\left(\lambda \int_0^t X(s)ds\right). \end{aligned} \tag{7}$$

The Lévy inflow L and the Poisson discharge 'trigger' Π – the system's sources of randomness – are independent processes.

Since the charging process L is Lévy, the resulting capacitor process X is *Markov*. Furthermore, X is a *regenerative* process: it *regenerates* at every discharge epoch. Indeed, if we denote by τ the capacitor's first discharge epoch, then the trajectory of the process X is the *concatenation* of independent and identically distributed copies of the truncated (and monotone increasing) trajectory $\{(s, X(s))\,|0 \leqslant s < \tau\}$. We henceforth refer to these truncated trajectories as *charging cycles*; to their duration τ as the *cycle duration*; and to their endpoints $X(\tau-)$ as the *cycle peaks*.

2.3. 'Standard' vs 'Capacitor' Ornstein–Uhlenbeck Dynamics

To conclude this section, let us compare the capacitor dynamics of Eq. (7) to corresponding 'standard' Lévy-driven Ornstein–Uhlenbeck dynamics. To that end, consider the 'standard' Lévy-driven Ornstein–Uhlenbeck system $Y = (Y(t))_{t \geqslant 0}$ governed by the stochastic dynamics

$$Y(dt) = -\lambda Y(t)dt + L(dt). \tag{8}$$

Both the dynamics given in Eqs. (7) and (8) are: (i) Markovian and taking place on the non-negative half-line; (ii) driven by the one-sided Lévy inflow L which is 'fed' additively and is 'pushing' the systems towards $+\infty$; (iii) subject to a retrieving force 'pushing' the systems back towards the origin 0.

[3]That is; if L is α-selfsimilar then the probability tails of its increments decay algebraically following a power-law of order α.

In the dynamics (7) the rate, at time t, of the retrieving force is $\lambda X(t)$. In the dynamics (8) the magnitude, at time t, of the retrieving force (the system's 'drift') is $\lambda Y(t)$. Hence, in both systems the intensity of the retrieving forces is linear (with coefficient λ) in the system's level. However, the *mechanism* of the retrieving forces is fundamentally different: in the 'standard' dynamics (8) the force acts *continuously* – pushing the system *smoothly* towards the origin, whereas in the capacitor dynamics (7) the force acts *discontinuously* – pushing the system *abruptly* towards the origin. In other words, in the dynamics (7) the retraction to the origin takes place at discrete time epochs (rather than continuously) and happens 'in a bang' (rather than smoothly).

Since, in both systems, the intensity of the retrieving force is *linear* in the system's level and the inflow process is 'fed' *additively* – we refer to the motion governed by Eq. (8) as "standard Lévy-driven Ornstein–Uhlenbeck dynamics", and coin the motion governed by Eq. (7) "*Lévy-charged Ornstein–Uhlenbeck capacitor dynamics*".

3. THE TIME-TO-DISCHARGE

Assume that the capacitor is at the level x ($x \geqslant 0$), and set τ_x to be the '*time-to-discharge*', i.e., the first passage time to the level 0. With no loss of generality we take $X(0) = x$ and hence

$$\tau_x = \inf\{t > 0 | X(t) = 0\}.$$

What is the distribution of τ_x?

Well, *given* the trajectory of the inflow process $L = (L(t))_{t \geqslant 0}$, we have

$$\begin{aligned}
&\mathbf{P}(\tau_x > t | L) \\
&\quad = \exp\left\{-\int_0^t r\,(x + L(s))\,ds\right\} \\
&\quad = \exp\left\{-\lambda x t - \int_0^t \lambda(t-s) L(ds)\right\}.
\end{aligned}$$

Hence, using conditioning and Eq. (6) we obtain

$$\begin{aligned}
&\mathbf{P}(\tau_x > t) \\
&\quad = \mathbf{E}[\mathbf{P}(\tau_x > t | L)] \\
&\quad = \exp\left\{-\lambda x t - \int_0^t \phi(\lambda(t-s))ds\right\}.
\end{aligned}$$

And, using Eq. (5), we can therefore conclude that

$$\mathbf{P}(\tau_x > t) = \exp\left\{-\lambda x t - \frac{1}{\lambda}\Phi(\lambda t)\right\}. \tag{9}$$

The cycle duration: Charging cycles always start from the level $x=0$ and hence their length – the *cycle duration* τ – equals τ_0. Thus, using Eq. (9), we obtain that the probability tails of the cycle duration τ $(=\tau_0)$ are given by

$$\mathbf{P}(\tau > t) = \exp\left\{-\frac{1}{\lambda}\Phi(\lambda t)\right\}. \tag{10}$$

Equation (10), in turn, implies that the mean cycle duration is given by

$$\mathbf{E}[\tau] = \frac{1}{\lambda}\int_0^\infty \exp\left\{-\frac{1}{\lambda}\Phi(u)\right\} du. \tag{11}$$

Note that the mean cycle duration $\mathbf{E}[\tau]$ is always *finite*! (Indeed; the characteristic Laplace exponent ϕ starts at the origin, and is monotone increasing and concave. This implies that $\Phi(u) \geqslant cu$ for all $u \geqslant u_0$, where c and u_0 are positive constants. This, in turn, implies that the integral on the right-hand side of Eq. (11) is finite.)

3.1. Hazard Functions

Differentiating the *logarithm* of Eq. (9) yields

$$H_x(t) := \frac{F_x'(t)}{1 - F_x(t)} = \lambda x + \phi(\lambda t), \tag{12}$$

where $F_x(t)$, $t \geqslant 0$, denotes the distribution function of τ_x $(F_x(t) = \mathbf{P}(\tau_x \leqslant t))$. The function $H_x(t)$ is known – in applied probability and reliability theory - as the *Failure Rate* or *Hazard Function* of the random variable τ_x.(14–16) Its probabilistic interpretation is

$$\lim_{h\to 0} \frac{1}{h}\mathbf{P}(t < \tau_x \leqslant t + h | \tau_x > t) = H_x(t), \tag{13}$$

i.e., $H_x(t)$ is the local *conditional* rate of discharge, at time t, *given* that there was no discharge up to time t. Since the characteristic Laplace exponent $\phi(\omega)$ is monotone increasing – so are the Hazard Functions $H_x(t)$

(for all $x \geqslant 0$). This implies that the distributions of the random variables τ_x, $x \geqslant 0$, satisfy the – so called – *Increasing Failure Rate* (IFR) property. For a detailed treatment of distributions with IFR we refer the reader to ref. 14. In particular, for τ_0 (=the cycle duration) we have $H_0(t) = \phi(\lambda t)$ and hence we can conclude that: *up to the scaling factor λ, the Hazard Function of the cycle duration equals the characteristic Laplace exponent of the Lévy inflow.*

3.2. Examples

We conclude this section with the distributions of cycle duration τ for the Lévy inflow examples given in the end of Section 3.1.

1. Compound Poisson inflow with exponential jumps ($\phi(\omega) = a\omega/(\nu+\omega)$; $a, \nu > 0$)

$$\mathbf{P}(\tau > t) = \frac{(1+at/c)^c}{\exp\{at\}}, \tag{14}$$

where $c = a\nu/\lambda$.

2. Gamma inflow ($\phi(\omega) = a\ln(1+\omega/\nu)$; $a, \nu > 0$)

$$\mathbf{P}(\tau > t) = \frac{\exp\{at\}}{\left(1+\frac{at}{c}\right)^{c+at}}, \tag{15}$$

where $c = a\nu/\lambda$.

3. Selfsimilar and deterministic inflows ($\phi(\omega) = a\omega^\alpha$; $a > 0$, $0 < \alpha \leqslant 1$)

$$\mathbf{P}(\tau > t) = \exp\{-(t/c)^{1+\alpha}\}, \tag{16}$$

where $c = (a\lambda^\alpha/(1+\alpha))^{-1/(1+\alpha)}$. For these inflows we also have a simple moment structure ($m > 0$)

$$\mathbf{E}[\tau^m] = \Gamma\left(1 + \tfrac{m}{1+\alpha}\right) c^m .$$

In the first two examples the resulting probability distributions of the cycle duration (given in Eqs. (14) and (15)) are rather peculiar and exceptional – although the Lévy inflow, in both cases, was fairly simple. For selfsimilar Lévy inflows the resulting probability distribution of the cycle duration is an *enhanced exponential*. Observe the counter-intuitive behavior implied form the structure of the probability tails given in Eq. (16):

The *wilder* the charge inflow (i.e., the *smaller* the self-similarity index α) – the *slower* the decay of the cycle duration. In particular, the cycle duration with slowest decay is obtained in the limit $\alpha \to 0$, yielding the *exponential* distribution. On the other hand, the cycle duration with fastest decay is obtained at the parameter value $\alpha = 1$ (corresponding to the case of deterministic inflows), yielding a probability distribution which follows, asymptotically, a *Gaussian* decay.

4. CYCLE PEAKS

In this section, we explore the behavior of the cycle peak $X(\tau-)$. To this end we use Lemma A.1 (see Appendix A) which implies that ($\omega \geqslant 0$)

$$\begin{aligned} &\lim_{h\to 0} \tfrac{1}{h}\mathbf{E}[\exp\{-\omega X(\tau-)\}\cdot \mathbf{I}\{t<\tau\leqslant t+h\}] \\ &= \exp\left\{-\tfrac{1}{\lambda}\left(\Phi(\omega+\lambda t)-\Phi(\omega)\right)\right\}\left(\phi(\omega+\lambda t)-\phi(\omega)\right), \end{aligned} \tag{17}$$

where $\mathbf{I}\{E\}$ denotes the indicator function of the event E.

Using Eq. (17) we thus obtain that

$$\mathbf{E}[\exp\{-\omega X(\tau-)\}] = \int_0^\infty g(t;\omega)dt,$$

where $g(t;\omega)$ is a shorthand notation of the 'density' appearing on the right hand side of Eq. (17). A calculation of the last integral yields the Laplace transform of the cycle peak $X(\tau-)$ ($\omega \geqslant 0$)

$$\mathbf{E}[\exp\{-\omega X(\tau-)\}] = 1 - \tfrac{\phi(\omega)}{\lambda}\exp\left\{\tfrac{1}{\lambda}\Phi(\omega)\right\}\int_\omega^\infty \exp\left\{-\tfrac{1}{\lambda}\Phi(u)\right\}du\,. \tag{18}$$

In the vicinity of $\omega = 0$ the Laplace transform given in Eq. (18) admits a particularly simple form: computing the limit of Eq. (18) as $\omega \to 0$, while making use of Eq. (11), gives

$$\mathbf{E}[\exp\{-\omega X(\tau-)\}] \underset{\omega\to 0}{\sim} 1 - \mathbf{E}[\tau]\,\phi(\omega). \tag{19}$$

Equation (19), in turn, implies that

(i) The inflow process L has a finite mean if and only if the cycle peak $X(\tau-)$ has a finite mean. Specifically

$$\mathbf{E}[X(\tau-)]=\mathbf{E}[L(1)]\,\mathbf{E}[\tau]. \tag{20}$$

(ii) Due to Karamata's Tauberian theorem (see, e.g., ref. 17), the distribution of the inflow process L is heavy-tailed with tails of order $0<\alpha<1$ if and only if the distribution of the cycle peak $X(\tau-)$ is such. Specifically

$$\mathbf{P}(L(1)>l) \underset{l\to\infty}{\sim} \frac{a}{l^{\alpha}} \quad \Leftrightarrow \quad \mathbf{P}(X(\tau-)>x) \underset{x\to\infty}{\sim} \frac{a\mathbf{E}[\tau]}{x^{\alpha}}. \tag{21}$$

5. CHARGING CYCLES

In this section, we study the *conditional* trajectory of the capacitor process during a charging cycle $\{X(s)|0\leqslant s<\tau\}$, *given* that $\tau=T$. To this end we use Lemma A.2 (see Appendix A) which gives the infinite-dimensional Laplace transform of the cycle's trajectory: for any 'nice' test function $\varphi(s)$, $0\leqslant s<T$, we have

$$\mathbf{E}\left[\exp\left\{-\int_0^T \varphi(s)X(ds)\right\}|\tau=T\right]=\frac{G(T;\varphi)}{G(T;0)}, \tag{22}$$

where the functional G is given by

$$G(T;\varphi)=\exp\left\{-\int_0^T \phi\left(\varphi(s)+\lambda(T-s)\right)ds\right\}\int_0^T \phi'\left(\varphi(s)+\lambda(T-s)\right)ds\,.$$

The explicit computation of $G(T;\varphi)$, for a general test function φ, is not possible. However, in (at least) two special cases – empirical averages and conditional trendlines – explicit analytical formulae can be derived.

5.1. Empirical Average

Let $\bar{X}=\left(\bar{X}(t)\right)_{t\geqslant 0}$ denote the *empirical average* of the capacitor process $X=(X(t))_{t\geqslant 0}$, i.e.,

$$\bar{X}(t)=\frac{1}{t}\int_0^t X(s)ds.$$

Then, taking $\varphi(s)=\theta(1-s/T)$, θ being a non-negative parameter, gives $\int_0^T \varphi(s)X(ds)=\theta\bar{X}(T)$. Hence, using Eq. (22) yields the *conditional* Laplace transform of the empirical cycle average $\bar{X}(\tau)$, *given* that $\tau=T$

$$\mathbf{E}\left[\exp\{-\theta\bar{X}(T)\}|\tau=T\right]=\frac{\exp\left\{-T\frac{\Phi(\lambda T+\theta)}{\lambda T+\theta}\right\}\frac{\phi(\lambda T+\theta)}{\lambda T+\theta}}{\exp\left\{-T\frac{\Phi(\lambda T)}{\lambda T}\right\}\frac{\phi(\lambda T)}{\lambda T}}. \tag{23}$$

5.2. Conditional Trendline

Let $0\leqslant t<T$ and set $\mathbf{I}_t$ to be the indicator function of the interval $(0,t)$. Taking $\varphi(s)=\theta\mathbf{I}_t(s)$, θ being a non-negative parameter, and using Eq. (22) we arrive at

$$\begin{aligned}
&\mathbf{E}[X(t)|\tau=T]\\
&=\frac{-\partial}{\partial\theta}\mathbf{E}[\exp\{-\theta X(t)\}\,|\tau=T]\big|_{\theta=0}\\
&=\frac{-\partial}{\partial\theta}\mathbf{E}\left[\exp\left\{-\int_0^T\theta\mathbf{I}_t(s)X(ds)\right\}|\tau=T\right]\Big|_{\theta=0}\\
&=-G(T;0)^{-1}\cdot\frac{\partial}{\partial\theta}G(T;\theta\mathbf{I}_t)\big|_{\theta=0}.
\end{aligned}$$

Calculating $\frac{\partial}{\partial\theta}G(T;\theta\mathbf{I}_t)\big|_{\theta=0}$ we conclude that the *conditional* mean of $X(t)$, *given* that $\tau=T$, is:

$$\mathbf{E}[X(t)|\tau=T]=\frac{\phi(\lambda T)-\phi(\lambda(T-t))}{\lambda}-\frac{\phi'(\lambda T)-\phi'(\lambda(T-t))}{\phi(\lambda T)}. \tag{24}$$

From Eq. (24) it is straightforward to deduce that the *conditional trendline* function

$$m(t;T):=\mathbf{E}[X(t)|\tau=T]$$

satisfies the following properties:

(i) it starts from the origin: $m(0;T)=0$;

(ii) it is monotone increasing (in the variable t);

(iii) it converges to a finite value as $t\to T$ if and only if L has a finite mean – otherwise $\lim_{t\to T}m(t;T)=\infty$.

Note that even if the charge inflow L has *infinite* mean, the conditional mean $m(t;T)$ is nevertheless *finite* for all $0 \leqslant t < T$ (!), and diverges only at the limit $t \to T$.

5.3. The Conditional Distribution of the Cycle Peaks

Let us take a closer look at what happens at the end point $t = T$. Taking $\varphi(s) \equiv \theta$ (θ being a non-negative parameter) and calculating Eq. (22) yields the *conditional* Laplace transform of the cycle peak $X(\tau-)$, *given* that $\tau = T$

$$\begin{aligned} &\mathbf{E}[\exp\{-\theta X(T-)\}|\tau = T] \\ &= \exp\left\{-\tfrac{1}{\lambda}\left(\Phi(\lambda T+\theta) - \Phi(\lambda T) - \Phi(\theta)\right)\right\} \tfrac{\phi(\lambda T+\theta)-\phi(\theta)}{\phi(\lambda T)}. \end{aligned} \tag{25}$$

If the inflow L has finite mean (i.e., if $\phi'(0) < \infty$) then differentiating Eq. (25) and taking $\theta = 0$ yields $m(T;T)$. However, if L has *infinite* mean then

$$\mathbf{E}[\exp\{-\theta X(T-)\}|\tau = T] \underset{\theta \to 0}{\sim} 1 - \frac{\phi(\theta)}{\phi(\lambda T)},$$

which, in turn (due to Karamata's Tauberian theorem), implies that L is heavy-tailed with tails of order $0 < \alpha < 1$ if and only if the *conditional* distribution of the cycle peak is such. Specifically

$$\mathbf{P}(L(1) > l) \underset{l \to \infty}{\sim} \frac{a}{l^{\alpha}} \quad \Leftrightarrow \quad \mathbf{P}(X(T-) > x|\tau = T) \underset{x \to \infty}{\sim} \frac{a/\phi(\lambda T)}{x^{\alpha}}.$$

5.4. Conditioning on the Event $\{\tau > T\}$

Analogous results can be derived given that $\tau > T$ (rather than $\tau = T$). Indeed, the counterparts of Eqs. (22)–(25) are, respectively

$$\mathbf{E}\left[\exp\left\{-\int_0^T \varphi(s) X(ds)\right\}|\tau > T\right] = \exp\left\{\tfrac{1}{\lambda}\Phi(\lambda T) - \int_0^T \phi\left(\varphi(s) + \lambda(T-s)\right) ds\right\},$$

$$\mathbf{E}\left[\exp\{-\theta \bar{X}(T)\}|\tau > T\right] = \exp\left\{-T\left(\tfrac{\Phi(\lambda T+\theta)}{\lambda T+\theta} - \tfrac{\Phi(\lambda T)}{\lambda T}\right)\right\},$$

$$\mathbf{E}[X(t)|\tau > T] = \tfrac{1}{\lambda}\left\{\phi(\lambda T) - \phi(\lambda(T-t))\right\},$$

$$\mathbf{E}[\exp\{-\theta X(T)\}|\tau > T] = \exp\left\{-\tfrac{1}{\lambda}\left(\Phi(\lambda T+\theta) - \Phi(\lambda T) - \Phi(\theta)\right)\right\}.$$

In particular, we obtain that the *conditional* mean of $X(T)$, *given* that $\tau > T$, is given by the simple formula

$$\mathbf{E}[X(T)|\tau > T] = \frac{1}{\lambda}\phi(\lambda T).$$

Again, this conditional mean is *finite* for all Lévy inflows L! Furthermore, using Eqs. (12) and (13) we obtain the following connection between the Hazard Function of the cycle duration τ and conditional mean $\mathbf{E}[X(T)|\tau > T]$

$$\lim_{h\to 0}\frac{1}{h}\mathbf{P}(T < \tau \leqslant T + h|\tau > T) = \lambda\mathbf{E}[X(T)|\tau > T].$$

5.5. Example: Selfsimilar Lévy Inflows

We conclude this section with the example of a selfsimilar Lévy inflows (i.e., $\phi(\omega) = a\omega^{\alpha}$, with $a > 0$ and $0 < \alpha < 1$) for which the above formulae yield

$$\begin{aligned}
\mathbf{E}\left[\exp\{-\theta\bar{X}(T)\}|\tau = T\right] &= \exp\left\{\frac{-aT}{1+\alpha}\left((\lambda T+\theta)^{\alpha} - (\lambda T)^{\alpha}\right)\right\}\left(\frac{\lambda T}{\lambda T+\theta}\right)^{1-\alpha},\\
\mathbf{E}\left[\exp\{-\theta\bar{X}(T)\}|\tau > T\right] &= \exp\left\{\frac{-aT}{1+\alpha}\left((\lambda T+\theta)^{\alpha} - (\lambda T)^{\alpha}\right)\right\},\\
\mathbf{E}[X(t)|\tau = T] &= \frac{aT^{\alpha}}{\lambda^{1-\alpha}}\left(1-\left(1-\frac{t}{T}\right)^{\alpha}\right) + \frac{\alpha}{\lambda T}\left(\left(1-\frac{t}{T}\right)^{\alpha-1} - 1\right),\\
\mathbf{E}[X(t)|\tau > T] &= \frac{aT^{\alpha}}{\lambda^{1-\alpha}}\left(1-\left(1-\frac{t}{T}\right)^{\alpha}\right).
\end{aligned}$$

6. EVOLUTION AND EQUILIBRIUM

In this last section, we turn to study the stationary equilibrium and evolution of the Lévy-charged Ornstein–Uhlenbeck capacitor system (7). We tackle this issue using two different methodological approaches: a *Renewal Theory* approach and a *Markov Processes* approach.

6.1. Equilibrium: Renewal Theory Analysis

Consider a capacitor system at equilibrium and assume, with no loss of generality, that the system initiated at $t = -\infty$. Let X_{∞} denote the system's stationary equilibrium level.

The system's discharge epochs form a renewal process with inter-discharge intervals of length τ. Hence, the time elapsing from an *arbitrary* time point (say $t=0$) till the next discharge epoch is *not* τ, but rather, the *residual lifetime* τ_{res} of τ – whose distribution tails are given by (see, e.g., ref. 18)

$$\mathbf{P}(\tau_{\text{res}} > t) = \frac{1}{\mathbf{E}[\tau]} \int_t^\infty \mathbf{P}(\tau > u)\, du. \tag{26}$$

Combining Eq. (26) and Eqs. (10) and (11) together we obtain that

$$\mathbf{P}(\tau_{\text{res}} > t) = \frac{\int_{\lambda t}^\infty \exp\{-1/\lambda \Phi(u)\}\, du}{\int_0^\infty \exp\{-1/\lambda \Phi(u)\}\, du}. \tag{27}$$

On the other hand, the system's level at an arbitrary time point (say $t=0$) is X_∞ and hence, using Eq. (9), we obtain that

$$\begin{aligned} &\mathbf{P}(\tau_{\text{res}} > t) \\ &= \textstyle\int_0^\infty \mathbf{P}(\tau_x > t)\, F_{X_\infty}(dx) \\ &= \textstyle\int_0^\infty \exp\left\{-\lambda x t - \frac{1}{\lambda}\Phi(\lambda t)\right\} F_{X_\infty}(dx) \\ &= \exp\left\{-\frac{1}{\lambda}\Phi(\lambda t)\right\} \mathbf{E}[\exp\{-\lambda t X_\infty\}], \end{aligned} \tag{28}$$

where $F_{X_\infty}(x)$, $x \geqslant 0$, denotes the distribution function of X_∞. Equating Eqs. (27) and (28), while taking $\omega = \lambda t$, we conclude that:

The Laplace transform of the system's stationary equilibrium distribution X_∞ is given by ($\omega \geqslant 0$):

$$\mathbf{E}[\exp\{-\omega X_\infty\}] = \exp\left\{\frac{1}{\lambda}\Phi(\omega)\right\} \frac{\int_\omega^\infty \exp\{-1/\lambda \Phi(u)\}\, du}{\int_0^\infty \exp\{-1/\lambda \Phi(u)\}\, du}. \tag{29}$$

Furthermore, from Eq. (29) we immediately obtain that

$$\mathbf{E}[X_\infty] = \frac{1}{\int_0^\infty \exp\left\{-\frac{1}{\lambda}\Phi(u)\right\} du} = \frac{1}{\lambda \mathbf{E}[\tau]}, \tag{30}$$

which is *finite* for all Lévy inflows L!

Last, we note that combining Eqs. (18) and (29) together yields the following functional connection between the Laplace transform of the stationary equilibrium X_∞ and the Laplace transform of the cycle peak $X(\tau-)$

$$\mathbf{E}[\tau]\phi(\omega)\cdot\mathbf{E}[\exp\{-\omega X_\infty\}]+\mathbf{E}[\exp\{-\omega X(\tau-)\}]=1.$$

6.2. 'Standard' *vs* 'Capacitor' Ornstein–Uhlenbeck Dynamics

It is interesting to compare the equilibrium behavior of Ornstein–Uhlenbeck capacitor dynamics (7) to the equilibrium behavior of the corresponding 'standard' Ornstein–Uhlenbeck dynamics (8). The stationary equilibrium level of the 'standard' Ornstein–Uhlenbeck system (8) – denote it by Y_∞ – is given by the Laplace transform (see, e.g., ref. 19)

$$\mathbf{E}[\exp\{-\omega Y_\infty\}]=\exp\left\{-\frac{1}{\lambda}\int_0^\omega\frac{\phi(u)}{u}du\right\}.$$

This implies that $\mathbf{E}[Y_\infty]=\lambda^{-1}\mathbf{E}[L(1)]$, in the finite-mean case, and

$$\mathbf{P}(L(1)>l)\underset{l\to\infty}{\sim}\frac{a}{l^\alpha}\quad\Leftrightarrow\quad\mathbf{P}(Y_\infty>y)\underset{x\to\infty}{\sim}\frac{a/(\lambda\alpha)}{y^\alpha}$$

in the heavy-tailed case $(0<\alpha<1)$.

The difference between the equilibrium behavior of the two Ornstein–Uhlenbeck systems, (7) and (8), is hence sharp. The 'standard' Ornstein–Uhlenbeck system *preserves* the mean and tail statistics of the inflow L. The capacitor Ornstein–Uhlenbeck system, however, 'smoothens out' the inflow in a much more powerful way: no matter how 'heavy' the probability tails of the inflow are – the equilibrium level of the system always possesses a finite mean!

This is well illustrated by the example of selfsimilar Lévy inflows where $\phi(\omega)=a\omega^\alpha$ (with $a>0$ and $0<\alpha<1$). The log-Laplace transforms of $L(1)$ and Y_∞ are given, respectively, by

$$-\ln\mathbf{E}[\exp\{-\omega L(1)\}]=a\omega^\alpha$$

and

$$-\ln\mathbf{E}[\exp\{-\omega Y_\infty\}]=\frac{a}{\lambda\alpha}\omega^\alpha.$$

Hence, the equilibrium level Y_∞ retains – up to a multiplicative constant – the same distributional structure of the underlying Lévy driver L. However, the distribution of equilibrium level of the capacitor Ornstein–Uhlenbeck system differs dramatically. Indeed, calculating Eq. (29) for $\phi(\omega)=a\omega^\alpha$ gives the log-Laplace transform

$$-\ln \mathbf{E}[\exp\{-\omega X_\infty\}]=\ln\left(\Gamma_\alpha(c\omega^{1+\alpha})\right)-c\omega^{1+\alpha},$$

where $c=a/\{\lambda(1+\alpha)\}$ and where

$$\Gamma_\alpha(s):=\frac{1}{\Gamma(1/1+\alpha)}\int_s^\infty \exp\{-u\}u^{1/(1+\alpha)-1}du \quad (s\geqslant 0).$$

The 'taming' of 'standard' Lévy-driven Ornstein–Uhlenbeck systems (8) is investigated in ref. 19. Namely, the following question is addressed: how can a system (8) driven by a heavy-tailed Lévy noise be 'restrained' so that the distribution of its equilibrium level will be more 'tamed' and 'well-behaved' than the distribution of its driving Lévy noise? The solution given (and analyzed) in ref. 19 is to use a *non-linear* – rather than linear – retrieving force. This results in a Lévy-driven Langevin system of the form

$$Y(dt)=-f(Y(t))dt+L(dt),$$

where $f(y)$ is the magnitude of the retrieving force. Hence the 'taming' is obtained by the use of a *non-linear*, *continuous*, and *deterministic* retrieving force.

The 'capacitor-type' restraining ('taming') mechanism explored in this manuscript is radically and oppositely different: it is *linear*, *discontinuous*, and *stochastic*. Furthermore, this restraining mechanism is even more powerful than the Langevin restraining mechanism. Indeed, if the driving Lévy noise is heavy-tailed then 'Langevin restraining' would yield a finite-mean equilibrium level only if the retrieving force $f(y)$ is strong enough to overcome the noise. However, as we demonstrated above, 'capacitor restraining' *always* results in finite-mean equilibrium level – no matter how 'wild' the Lévy noise is.

6.3. Evolution: Markovian Analysis

Given a 'nice' test function $\varphi(x)$, $x\geqslant 0$, Dynkin's formula (see, e.g., ref. 20) asserts that

$$\mathbf{E}[\varphi(X(t))]=\mathbf{E}[\varphi(X(0))]+\int_0^t \mathbf{E}[(\mathcal{L}\varphi)(X(s))]ds, \tag{31}$$

where $\mathcal{L}$ is the *infinitesimal generator* of the Markov process $X=(X(t))_{t\geqslant 0}$

$$(\mathcal{L}\varphi)(x)=\lim_{h\to 0}\frac{\mathbf{E}[\varphi(X(h))|X(0)=x]-\varphi(x)}{h}.$$

Since our system is governed by the transitions (3), it is straightforward to deduce that

$$(\mathcal{L}\varphi)(x)=\lambda x(\varphi(0)-\varphi(x))+\lim_{h\to 0}\frac{\mathbf{E}[\varphi(x+L(h))]-\varphi(x)}{h}$$

the second term on the right-hand side being the infinitesimal generator[4] of the Lévy inflow L. In particular, if we take $\varphi_\omega(x)=\exp\{-\omega x\}$ (with $\omega\geqslant 0$) and use Eq. (4) we have

$$(\mathcal{L}\varphi_\omega)(x)=\lambda x(1-\exp\{-\omega x\})-\phi(\omega)\exp\{-\omega x\}.$$

Hence, setting $V(t;\omega):=\mathbf{E}[\exp\{-\omega X(t)\}]$ to be the Laplace transform of $X(t)$, and $m(t):=\mathbf{E}[X(t)]$ to be the mean of $X(t)$, we obtain that

$$\mathbf{E}[(\mathcal{L}\varphi_\omega)(X(s))]=\lambda m(t)+\lambda\frac{\partial V}{\partial\omega}(t,\omega)-\phi(\omega)V(t;\omega).$$

Finally, using Dynkin's formula (31), we can conclude that $V(t;\omega)$ satisfies the partial differential equation $(t,\omega>0)$

$$\left(\frac{\partial V}{\partial\omega}-\frac{1}{\lambda}\frac{\partial V}{\partial t}\right)(t,\omega)-\frac{\phi(\omega)}{\lambda}V(t;\omega)=-m(t), \tag{32}$$

with the initial condition $V(0;\omega)=\mathbf{E}[\exp\{-\omega X(0)\}]$ and the boundary condition $V(t;0)=1$.

The solution of the partial differential equation (32) is given by the convolution

$$V(t;\omega)=\exp\left\{\tfrac{1}{\lambda}\Phi(\omega)\right\}\left(1-\int_0^\omega\exp\left\{-\tfrac{1}{\lambda}\Phi(u)\right\}m\left(t+\tfrac{\omega-u}{\lambda}\right)du\right). \tag{33}$$

[4]The infinitesimal generator of the Lévy inflow L admits the representation

$$\lim_{h\to 0}\frac{\mathbf{E}[\varphi(x+L(h))]-\varphi(x)}{h}=\int_0^\infty(\varphi(x+u)-\varphi(x))\,J(du),$$

where $J(du)$ is the *jump measure* (Lévy measure) of the Lévy inflow L. For further details the readers are referred to Skorokhod[(7)].

In particular, taking $t \to \infty$ in Eq. (33) gives

$$\mathbf{E}[\exp\{-\omega X_\infty\}] = \exp\left\{\tfrac{1}{\lambda}\Phi(\omega)\right\}\left(1 - \mathbf{E}[X_\infty]\int_0^\omega \exp\left\{-\tfrac{1}{\lambda}\Phi(u)\right\} du\right), \quad (34)$$

which, in turn, yields the stationary equilibrium solution (29) (indeed; taking $\omega \to 0$ gives Eq. (30) which, after substituting back into Eq. (34), yields Eq. (29)).

6.4. Mean and Auto-correlation

From Eq. (33) the following implicit convolution equation for the mean function $m(t) = \mathbf{E}[X(t)]$ is obtained:

$$\begin{aligned} &\int_0^t \exp\left\{-\tfrac{1}{\lambda}\Phi(\lambda s)\right\} m(t-s)\, ds \\ &= 1 - \exp\left\{-\tfrac{1}{\lambda}\Phi(\lambda t)\right\} \mathbf{E}[\exp\{-\lambda t X(0)\}]. \end{aligned} \quad (35)$$

Indeed; taking $t = 0$ in Eq. (33), and then setting $\omega/\lambda = t$ and using the change of variable $u/\lambda = s$ yields Eq. (35).

Furthermore, an analogous implicit convolution equation for the system's auto-correlation function $C(t) = \mathbf{E}[X(0)X(t)]$, *at equilibrium*, is derived

$$\begin{aligned} &\int_0^t \exp\left\{-\tfrac{1}{\lambda}\Phi(\lambda s)\right\} C(t-s)\, ds \\ &= \mathbf{E}[X_\infty]\left(1 - \exp\left\{-\tfrac{1}{\lambda}\Phi(\lambda t)\right\} + \phi(\lambda t)\int_t^\infty \exp\left\{-\tfrac{1}{\lambda}\Phi(\lambda s)\right\} ds\right). \end{aligned} \quad (36)$$

We explain

Set $m_x(t) = \mathbf{E}[X(t)|X(0) = x]$ and note that: (i) using conditioning, the auto-correlation function $C(t)$ is given by

$$\begin{aligned} &\mathbf{E}[X(0)X(t)] \\ &= \mathbf{E}[X(0)\mathbf{E}[X(t)|X(0) = x]] \\ &= \int_0^\infty x m_x(t)\, F_{X_\infty}(dx), \end{aligned} \quad (37)$$

where $F_{X_\infty}(\cdot)$ denotes the distribution function of the stationary equilibrium level X_∞; (ii) using Eq. (35) we have

$$\int_0^t \exp\left\{-\tfrac{1}{\lambda}\Phi(\lambda s)\right\} m_x(t-s)\, ds = 1 - \exp\left\{-\tfrac{1}{\lambda}\Phi(\lambda t)\right\} \exp\{-\lambda t x\}. \quad (38)$$

Now; multiplying Eq. (38) by x and integrating with respect to the distribution function $F_{X_\infty}(\cdot)$ gives

$$\begin{aligned}&\int_0^t \exp\left\{-\tfrac{1}{\lambda}\Phi(\lambda s)\right\} C\,(t-s)\,ds \\ &= \mathbf{E}[X_\infty] + \exp\left\{-\tfrac{1}{\lambda}\Phi(\omega)\right\} \tfrac{\partial}{\partial\omega}\mathbf{E}[\exp\{-\omega X_\infty\}]\Big|_{\omega=\lambda t}.\end{aligned} \tag{39}$$

Finally, calculating the right hand side of Eq. (39) (using Eq. (29)), yields Eq. (36).

7. CONCLUSIONS

We introduced a model of stochastic capacitor systems with random charging and discharging mechanisms. The charging mechanism is an inflow with stationary, independent, and non-negative valued increments – i.e., a one-sided Lévy process (Lévy subordinator). The discharging mechanism is Markovian, with a rate depending linearly on the capacitor's level. Discharging drops the capacitor's level to zero, right after which the stochastic charging cycle begins anew.

Since the inflow is Lévy, and since the Markovian rate is linear (in the capacitor's level), we coined these systems "*Lévy-charged Ornstein–Uhlenbeck capacitors*". This is in analogy to 'standard' Lévy-driven Ornstein–Uhlenbeck systems: systems with a Lévy inflow as above, and with a deterministic retrieving mechanism – acting continuously in time and 'pushing' the system back to the origin – whose magnitude is linear in the system's level.

Our investigation of Lévy-charged Ornstein–Uhlenbeck capacitors began with the computation of the distribution of the capacitor's time-to-discharge (given the system's initial charging level). We then turned to study various random quantities associated with the capacitors' charging cycles, including their: duration; peak height; empirical average; and, conditional trendline and peak (conditioned on the duration of the charging cycle). We concluded with the analysis of the capacitors' stationary equilibrium system-level. In our investigation we used probabilistic techniques from the theories of Lévy, Markov, and renewal processes.

The study unveiled two rather unexpected features of Lévy-charged Ornstein–Uhlenbeck capacitor systems

- The distribution of the cycle duration can turn out to be peculiar and exceptional (even in cases where the underlying Lévy inflow is simple), and its behavior can be counterintuitive (as in the case of selfsimilar Lévy inflow).

• The stationary equilibrium system-level always admits a finite mean – however 'wild' the Lévy inflow is. (The same is to be said about the conditional trendline during a charging cycle.)

The finite-mean property is a key feature of Lévy-charged Ornstein–Uhlenbeck capacitors, sharply distinguishing them from 'standard' Lévy-driven Ornstein–Uhlenbeck systems – for which the equilibrium system-level admits a finite mean if and only if the underlying Lévy inflow has a finite mean.

The finite-mean property also implies that it is possible to 'restrain' systems with 'wild' Lévy inflow using the proposed capacitor-type dynamics (where the retrieving mechanism is stochastic, discontinuous, and linear), rather than by using Langevin-type dynamics (where the retrieving mechanism is deterministic, continuous, and non-linear). Moreover, capacitor-type 'restraining' turns out to be more potent (and analytically more tractable) than Langevin-type 'restraining'.

APPENDIX A

We set

$$\mathbf{G}(t;\omega,\varphi)=\exp\left\{-\int_0^t\phi(\omega+\varphi(s)+\lambda(t-s))\,ds\right\}\int_0^t\phi'(\omega+\varphi(s)+\lambda(t-s))\,ds\,,$$

where $t,\omega\geqslant 0$ and $\varphi(s)$, $s\geqslant 0$, is a 'nice' test function.

Lemma A.1. $\forall\omega\geqslant 0$ and $\forall$'nice' test function $\varphi(s)$, $s\geqslant 0$, we have:

$$\lim_{h\to 0}\frac{1}{h}\mathbf{E}\left[\exp\left\{-\omega X(\tau-)-\int_0^{\tau-}\varphi(s)X(ds)\right\}\cdot\mathbf{I}\{t<\tau\leqslant t+h\}\right]=\lambda\mathbf{G}(t;\omega,\varphi)\,,$$

where $\mathbf{I}\{E\}$ denotes the indicator function of the event E.

Note that, in particular, Lemma A.1 implies that the probability density function of cycle duration τ is given by $\exp\left\{-\frac{1}{\lambda}\Phi(\lambda t)\right\}\phi(\lambda t)$ – in agreement with Eq. (10).

Proof. Fix $0<h<<1$, and note that

$$\begin{aligned}
&\mathbf{P}\,(t<\tau\leqslant t+h|L)\\
&\simeq\exp\left\{-\int_0^t r\,(L(s))\,ds\right\}r(L(t))h\\
&=\exp\left\{-\int_0^t\lambda(t-s)L(ds)\right\}\lambda L(t)h.
\end{aligned}$$

Hence, fixing (ω, φ) and using conditioning, we obtain

$$\begin{aligned}
&\mathbf{E}\left[\exp\left\{-\omega X(\tau-) - \int_0^{\tau-} \varphi(s)X(ds)\right\}\cdot \mathbf{I}\{t<\tau\leqslant t+h\}\right] \\
&= \mathbf{E}\left[\exp\left\{-\omega L(t) - \int_0^t \varphi(s)L(ds)\right\}\cdot \mathbf{P}\left(t<\tau\leqslant t+h|L\right)\right] \\
&\simeq \mathbf{E}\left[\exp\left\{-\omega L(t) - \int_0^t \varphi(s)L(ds) - \int_0^t \lambda(t-s)L(ds)\right\}\lambda L(t)\right]\cdot h \\
&= -\lambda\frac{\partial}{\partial\omega}\mathbf{E}\left[\exp\left\{-\omega L(t) - \int_0^t \varphi(s)L(ds) - \int_0^t \lambda(t-s)L(ds)\right\}\right]\cdot h \qquad \text{(A.1)} \\
&= -\lambda\frac{\partial}{\partial\omega}\mathbf{E}\left[\exp\left\{-\int_0^t \left(\omega+\varphi(s)+\lambda(t-s)\right)L(ds)\right\}\right]\cdot h \\
&= -\lambda\frac{\partial}{\partial\omega}\exp\left\{-\int_0^t \phi\left(\omega+\varphi(s)+\lambda(t-s)\right)ds\right\}\cdot h \\
&= \lambda\mathbf{G}(t;\omega,\varphi)\cdot h.
\end{aligned}$$

Finally; dividing Eq. (A.1) by h, and taking $h\to 0$ concludes the proof. ▮

An immediate corollary of Lemma A.1 is

Lemma A.2. $\forall\omega\geqslant 0$ and $\forall$'nice' test function $\varphi(s)$, $s\geqslant 0$, we have

$$\mathbf{E}\left[\exp\left\{-\omega X(\tau-) - \int_0^{\tau-} \varphi(s)X(ds)\right\}|\tau=t\right] = \frac{\mathbf{G}(t;\omega,\varphi)}{\mathbf{G}(t;0,0)}. \qquad \text{(A.2)}$$

Proof. Fix $0<h<<1$, and note that the conditional expectation

$$\mathbf{E}\left[\exp\left\{-\omega X(\tau-) - \int_0^{\tau-} \varphi(s)X(ds)\right\}|t<\tau\leqslant t+h\right]$$

equals

$$\frac{\mathbf{E}\left[\exp\left\{-\omega X(\tau-) - \int_0^{\tau-} \varphi(s)X(ds)\right\}\cdot \mathbf{I}\{t<\tau\leqslant t+h\}\right]}{\mathbf{P}\left(t<\tau\leqslant t+h\right)}.$$

Taking $h\to 0$ and using A.1 concludes the proof. ▮

REFERENCES

1. P. Bak, *How Nature Works: the Science of Self Organized Criticality* (Copernicus, 1996).
2. M. G. Rozman, M. Urbach, J. Klafter, and F. J. Elmer, *J. Phys. Chem. B* **102**:7924 (1988).
3. J. M. Carlson, J. S. Langer, and B. E. Shaw, *Rev. Mod. Phys.* **66**:657 (1994).
4. D. Brockmann and T. Geisel, *Neurocomputing* **643**: 32–33 (2000).
5. I. Eliazar and J. Klafter, *Physica A* **334**:1 (2004).
6. V. M. Zolotarev, *One-Dimensional Stable Distributions* (AMS, 1986).
7. A. V. Skorokhod, *Random Processes with Independent Increments* (Kluwer, 1991).
8. G. Samrodintsky and M. S. Taqqu, *Stable non-Gaussian Random Processes* (CRC Press, 1994).
9. J. Bertoin, *Lévy Processes* (Cambridge University Press, 1996).
10. K. Sato, *Lévy Processes and Infinitely Divisible Distributions* (Cambridge University Press, 1999).
11. V. V. Uchaikin and V. M. Zolotarev, *Chance and Stability, Stable Distributions and Their Applications* (V.S.P. Intl. Science, 1999).
12. O. E. Barndorff-Nielsen, T. Mikosch, and S. Resnic eds., *Lévy Processes* (Birkhauser, 2001).
13. P. Embrechts and M. Maejima, *Selfsimilar Processes* (Princeton University Press, 2002).
14. E. Barlow and F. Proschan, *Mathematical Theory of Reliability* (Classics in Applied Mathematics 17, reprint edition) (Society for Industrial & Applied Mathematics, 1996).
15. H. C. Tijms, *Stochastic Models: An Algorithmic Approach* (John Wiley & Sons, 1995).
16. S. M. Ross, *Introduction to Probability Models*, 8th ed. (Academic Press, 2002).
17. N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation* (Cambridge University Press, 1987).
18. S. M. Ross, *Applied Probability Models with Optimization Applications* (Holden-Day, 1970).
19. I. Eliazar and J. Klafter, *J. Stat. Phys.* **111** (314): 739 (2003).
20. B. Oksendal, *Stochastic Differential Equations* (Springer, 1995).